PHYSICAL REVIEW E **78**, 041902 (2008)

Conditions for correspondence between Hartree scattering and biological growth

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(Received 27 March 2008; revised manuscript received 25 June 2008; published 1 October 2008)

The population dynamics of a weakly scattering system can often be characterized by Hartree equations, and those of a living system by Lotka-Volterra (LV) equations. In principle, can the population statistics of the quantum scattering system follow those of the living system? The answer is yes, provided the interactive potentials of the Hartree equations are made equal, on a one-to-one basis, to corresponding fitness functions of the LV equations. Of course this correspondence can be achieved only if the requirements of the Hartree approximation are satisfied, including that the scatter occurs within the coherence time of the quantum system. Examples are given of Hartree systems that obey the population dynamics of required predator-prey systems.

DOI: 10.1103/PhysRevE.78.041902 PACS number(s): 87.17.Ee, 34.50.−s, 31.15.xr

I. BACKGROUND

The overall aim of this paper is to investigate whether quantum dynamics can play a substantial role in defining biological population dynamics. Of course, quantum dynamics are but one aspect of quantum theory. Other investigators [1–5] are considering the larger question of how all basic quantum effects [6]—including uncertainty relations, spin statistics, and quantum dynamics—relate to life functions. The first two effects are not addressed here.

Also, on the biological side, population dynamics are but one measure of a living system's functions. Other functions such as seeking and absorbing nutrients, mitosis, resistance to disease, etc., are intrinsic to life. This serves to limit the scope of the investigation. However, in fact, population dynamics are largely inclusive of these other functions, since population dynamics are defined by fitness values (defined below), and fitness values depend upon all attributes that contribute to fitness, including these other functions. This tends to broaden the scope of the approach.

The upshot is that this study limits attention to, specifically, the role that (a) quantum population dynamics can, in principle, play in forming (b) required biological population dynamics. For convenience, we use quantum dynamics in the Schrödinger picture, i.e., as they obey the Schrödinger wave equation (SWE). Hence the following central question: In principle, under what conditions can the population dynamics of a system of physical particles follow those of biology?

The reverse question—When does a biological system act quantum mechanically?—is also important. Both questions could be clarified by finding a correspondence principle connecting the two domains. This will center on finding conditions for which Hartree equations [6] of quantum mechanics go over into Lotka-Volterra (LV) equations of biological growth.

A. Caveats

As will be seen, the approach requires the existence of coherence during a fundamental scattering process. This strongly limits the scope of the approach, as discussed in Secs. II C and VII D.

It will be found that our physical application of the SWE leads to a transport equation of physical population growth, only a particular case of which is biological growth. The question of finding conditions for which biological growth in particular arises is briefly taken up in Sec. IX.

Also, the preceding central question avoids the issue that many nonquantum wave phenomena obey wave equations that can be put into the SWE form under suitable conditions, so that in fact any of these wave phenomena would suffice to achieve the required population dynamics. However, there is a possible rebuttal to this point. The approach will require use of the Hartree approximation, and, through generally all orders of its interaction potentials. A search of the literature seems to indicate that, among general wave phenomena, an SWE-Hartley approach applies uniquely to the particular phenomenon called quantum mechanics.

In starting from the SWE, we are ignoring relativistic- and particle-spin effects.

Finally, there are problems of accuracy in using the Hartree approximation in the first place. These are discussed in Secs. II B and II C.

B. Question of variable particle number

A conceivable theoretical obstacle to the development is that growth and depletion imply a changing number of particles whereas, to the contrary, conservation of particle number seems to be a hallmark of quantum physics [6]. However, in fact conservation of particle number only holds in the presence of a Hermitian Hamiltonian. This requires a real potential function, whereas we instead more generally use complex potentials. Many properties of scattering in the presence of complex potentials are taken up in [7,8].

II. HARTEE APPROACH

A living system contains many "particles," be they cells, trees or elephants. Therefore, the emulating quantum system must likewise contain multiple particles. The commonest approach to analyzing such quantum systems (in the presence of coherence) is the Hartree self-consistent-field [6] technique. We review it first, addressing as well its limitations, and then apply it to the problem at hand. The assumption of coherence is also discussed below.

A. Aim

The aim of the Hartree approach is to find the quantum dynamics of K particles $k=1,\ldots,K$ moving in a coherent system and under the influence of an exterior potential plus interaction potentials among them. In principle, the potentials can be any combination of real and imaginary quantities. The particle classes can be electrons, molecules, or others, depending upon application. The Hartee or Hartree-Fock approach has the virtue of being optimum from the point of view of a variational calculation [6] with a separable wave function.

B. Recent uses

The Hartree approach has had many successful applications, as the following brief list indicates. These are: in "generally describing atoms" [9] (summary statement); in describing fluid properties which are shared by QCD and cold atoms [10]; in analyzing charge oscillations in quantum dots [11]; in describing a meson field theory of nuclear ground states [12]; and in a detailed treatment of quantum fluctuations in nuclear matter, where two or more higher-order Hartree interaction terms must be used [13]. Most of its current applications appear to be to processes of physical chemistry, quantum dots, and nuclear interaction.

C. Limitations

As with any physical approximation, the Hartree has a limited domain of application. First, the resulting Hartree equations are a system of coupled SWEs which generally can only be solved iteratively. Therefore, convergence may be prohibitively slow, depending upon the particular potentials at hand. Next, the Hartree approach rapidly loses validity as the number of interacting particles increases, since dissipatory effects inevitably enter in, violating our premise of coherence. This is especially true if the system is open. Then the system density operator becomes nonlocal in time and, hence, dependent upon the past histories of external forces upon the particles [(a) in Ref. [14]]. However, our external forces are in fact zero [see Eq. (10)], helping to mitigate the problem. In general, the assumption of coherence strongly limits the scope of Hartree application (see Sec. VII D).

A well-known basic problem with the approach is that the particles are assumed to be (i) independent and (ii) indistinguishable. Assumption (i) neglects correlations between the positions of the particles, which may be considerable. Assumption (ii) is only approximately satisfied by the Hartree approach, but can be overcome by use of the Hartree-Fock approach. In fact the latter is within the scope of our approach, and briefly addressed in Sec. VII B. Assumption (i) also ignores correlations between electrons and nuclei in atoms, effectively inducing decoherence into the problem, thus violating our premise of coherence [(b) in Ref. [14]]. However, it does not generally rule it out, as discussed in Sec. VII D. An overview of limitations to Hartree applications that were encountered is in (c) of Ref. [14].

D. Algebraic expression

Let $\mathbf{r}=(x,y,z)$ denote an absolute position, and $\mathbf{r}_{jk}\equiv\mathbf{r}_k-\mathbf{r}_j$ denote a directed position, in the system. The dynamics are to be specified by the particle probability amplitude functions $\psi_j(\mathbf{r},t)$, and corresponding probabilities $p_j(\mathbf{r},t)=|\psi_j(\mathbf{r},t)|^2$. As mentioned above, any possible correlation in position of the particles is ignored, so that in effect their joint amplitude $\psi(\mathbf{r}_1,\mathbf{r}_2,\ldots\mathbf{r}_N,t)$ is assumed to be a product of the individual $\psi_j(\mathbf{r},t)$, $j=1,\ldots,K$.

The Hartree approach postulates that the kth particle moves in a net potential field that is the sum of its own local potential $v_k(\mathbf{r},t)$ plus its mean interaction potential $v_{jk}(\mathbf{r}_{jk},t)$ with all other particles $j\neq k$ in the system. Mean values are taken with respect to the jth particle's position probability density $|\psi_j(\mathbf{r},t)|^2$. Expressing this algebraically, the kth particle "sees" a total potential

$$V_k(\mathbf{r},t)=v_k(\mathbf{r},t)+\sum_j V_{jk}(\mathbf{r}_{jk},t), \quad \text{where } \mathbf{r}_{jj}\equiv\mathbf{r}, \tag{1}$$

$$V_{jk}(\mathbf{r}_{jk},t)\equiv\langle v_{jk}(\mathbf{r}_{jk},t)\rangle_j\equiv\int d\mathbf{r}_j|\psi_j(\mathbf{r}_j,t)|^2 v_{jk}(\mathbf{r}_{jk},t). \tag{2}$$

The indicated average $\langle\cdots\rangle_j$ is over all space $\mathbf{r}_j$, but not the time, which remains arbitrary but fixed. This is convenient since we are interested in the system behavior as a function of time. Generally all quantities V_k, v_k, V_{jk}, v_{jk}, $\psi_j(\mathbf{r}_j,t)$, K, etc., in the analysis depend on the time.

Note also that, since our ultimate interest is in biological particles, the usual restriction $j\neq k$ in the sum (1) has been lifted. That restriction arises in derivation of Eq. (1) by a variational principle [6], assuming the particles to be fermions (as in Sec. VII B). Here we more generally allow cases $j=k$ as well, where the particles are effectively bosons (Sec. VII C). This is an interesting parallel of the physics of particle populations with that of living populations.

III. SCHRÖDINGER EQUATION

The Hartree approach separates out the many-body problem, allowing a distinct SWE (Schrödinger wave equation) for the kth particle to be formed,

$$-\frac{\hbar^2}{2m_k}\nabla^2\psi_k+V_k(\mathbf{r},t)\psi_k=i\hbar\frac{\partial\psi_k}{\partial t}, \quad \psi_k\equiv\psi_k(\mathbf{r}_k,t), \tag{3}$$

where $V_k(\mathbf{r},t)$ is given by Eqs. (1) and (2). The SWE (3) thereby becomes the Hartree equation

$$-\frac{\hbar^2}{2m_k}\nabla^2\psi_k+\left(v_k(\mathbf{r},t)+\sum_j\int d\mathbf{r}_j|\psi_j(\mathbf{r}_j,t)|^2 v_{jk}(\mathbf{r}_{jk},t)\right)\psi_k = i\hbar\frac{\partial\psi_k}{\partial t}. \tag{4}$$

It is interesting that SWE-like differential equations have been shown, using a principle of extreme physical information (EPI) [15–20], to describe the dynamics of both quantum and classical system. This is through both the standard, quantum-level SWE and nonquantum versions that lack Planck's constant $\hbar$. This has been independently corroborated in [20], where it is shown that even the dynamics of classical effects can obey the mathematics of quantum theory. The use of Eqs. (1)–(4) will likewise lead to a classical SWE, i.e., lacking all dependence on $\hbar$. This classical-level SWE will be shown to give rise to Lotka-Volterra equations of transport, including those of biological growth.

It is to be noted that the interpretation of the SWE by EPI [15–17] automatically allows for time-dependent growth or depletion of particle populations, exactly as needed to describe biological systems. This follows because SWE-EPI solutions do not generally obey normalization over space at each time (this is a well-known effect, e.g., in cases of complex potentials [21]). Rather, they obey normalization over all space and all time. Therefore, at each time a generally different number of particles $N=N(t)$ can exist in the system, as required here.

IV. SOLUTION IN SPATIALLY HOMOGENEOUS MEDIUM

The most widely accepted equations of biological population change, the Lotka-Volterra equations (see below), do not depend upon space coordinates. Since these LV equations are our ultimate goal, assume that the potentials in the scattering medium are homogeneous in space; the kth particle sees only *a time*-dependent potential

$$V_k(\mathbf{r},t) \equiv V_k(t), \quad v_k(\mathbf{r},t) \equiv v_k(t), \quad v_{jk}(\mathbf{r}_{jk},t) \equiv v_{jk}(t). \tag{5}$$

The indicated time dependences are general. Using Eqs. (5), the SWE (3) becomes

$$-\frac{\hbar^2}{2m_k}\nabla^2\psi_k + V_k(t)\psi_k = i\hbar\frac{\partial\psi_k}{\partial t}, \quad \psi_k \equiv \psi_k(\mathbf{r},t) \tag{6}$$

with $V_k(t)$ obeying Eqs. (1) and (2) with the t dependence dropped. We also dropped the now unnecessary subscript k of $\mathbf{r}_k$. This wave equation (6) is next solved exactly.

By the usual separation of variables

$$\psi_k(\mathbf{r},t) \equiv R_k(\mathbf{r})T_k(t), \tag{7}$$

Eq. (6) separates into a free-particle wave equation

$$\left(\nabla^2 + \frac{2m_k\beta}{\hbar^2}\right)R_k(\mathbf{r}) = 0 \tag{8}$$

and a first-order equation

$$i\hbar\frac{dT_k(t)}{dt} = [\beta + V_k(t)]T_k(t) \tag{9}$$

that depends upon the potentials $V_k(t)$. Quantity β is the separation constant of the separation of variables approach, and is taken to be real. Note that this approach to solution does not assume conservation of energy E (the usual elementary case). Hence, the solution for $T(t)$ will not be the usual textbook answer $\exp(iEt/\hbar)$, with $\hbar$ Planck's constant.

The solution $R_k(\mathbf{r})$ to Eq. (8) is irrelevent to our purposes. Hence, we concentrate on the solution $T_k(t)$ to Eq. (9), which depends upon the form of $V_k(t)$.

V. SCATTERING WITH A COMPLEX POTENTIAL

The potential function $V_k(t)$ introduced in Eq. (5) is, so far, undefined. In most applications the potential is real. However, if it were real, the analysis that follows would lead only to familiar results—the equation of continuity of flow and the definitions of current density and current density (discussed at the end of Sec. VI).

Instead, consider the case where $V_k(t)$ is complex [7,8,23–25]. As is well known, such a potential when used in the SWE gives a non-Hermitian Hamiltonian and forms a wave function that intrinsically decays or grows in time [26]. Hence such a potential is often used in nuclear physics, e.g., to describe scattering events into inelastic channels where, as here, conservation of particle number is violated [24]. These particles are removed from the incident flux, and go on to form complex nuclei [27].

Complex biological systems, of central interest to us, turn out to obey like population effects. Thus, the nuclear inelastic channels in the preceding correspond to populations that dissipate or decrease in number, so that conservation of population number is violated. Likewise, the incident flux of particles is a new cohort of biological offspring. And the newly formed complex nuclei correspond to complex subecologies as adapted to local environments.

A. Particular complex potentials

Hence, consider the complex potentials

$$V_k(t) \equiv i\frac{\hbar}{2}G_k(t) - \beta, \quad v_k(t) \equiv i\frac{\hbar}{2}g_k(t) - \beta,$$
$$v_{jk}(t) \equiv i\frac{\hbar}{2}g_{jk}(t), \quad i = \sqrt{-1}. \tag{10}$$

The constant β and new potential functions $G_k(t)$, $g_k(t)$, and $g_{jk}(t)$ (the g's standing for growth, in later biological application) are real. In general, these functions depend upon known fixed constants c_k, c_{jk} which can represent evolutionary tactics in the corresponding biological system (discussed below). Since these potentials do not depend upon position, and assuming none of them are of magnetic origin, the particles are not acted upon by forces.

In (10), the imaginary part of $V_k(t)$ was expressed in units of $\hbar \equiv$ Planck's constant divided by 2π. Quantum energies are commonly expressed as linear in $\hbar$, following Planck's representation $\hbar\omega$ of the energy as linear in $\hbar$ (with ω a frequency). See, e.g., [28].

In the following, we show that the entirely classical equations (21) (below) of particle transport follows exactly from these assumptions.

B. Hartree scattering potentials→growth coefficients

Using Eqs. (10) in Eqs. (1) and (2) directly give

$$G_k(t) = g_k(t) + \sum_{j \neq k} G_{jk}(t), \tag{11}$$

$$G_{jk}(t) \equiv g_{jk}(t) \int d\mathbf{r} p_j(\mathbf{r},t), \tag{12}$$

$$p_k(\mathbf{r},t) \equiv |\psi_k(\mathbf{r},t)|^2 \equiv |R_k(\mathbf{r}) T_k(t)|^2. \tag{13}$$

The latter is by Eq. (7). Thus, by definitions (10), the original potentials V_k, v_k, V_{jk}, v_{jk} correspond one-to-one to the new potentials, as

$$V_k(\mathbf{r},t) \to G_k(t), \quad v_k(\mathbf{r},t) \to g_k(t), \quad V_{jk}(\mathbf{r}_{jk},t) \to G_{jk}(t),$$
$$v_{jk}(\mathbf{r}_{jk},t) \to g_{jk}(t). \tag{14}$$

These will be key to establishing the correspondence we seek between quantum dynamics and population dynamics.

VI. TRANSPORT EQUATIONS

Equations (10) show that our complex potentials $V_k(t)$, $v_k(t)$, and $v_{jk}(t)$ go as $\hbar$, a very small constant. This suggests very weak temporal effects. In fact, we next show that $\hbar$ does not contribute at all to the resulting growth law.

Plugging the first Eq. (10) into Eq. (9), and switching sides, gives

$$\frac{1}{2} G_k(t) T_k = \frac{dT_k}{dt}, \quad T_k = T_k(t), \tag{15}$$

the factor $\hbar$ having cancelled out. This is a handy, although not remarkable, result. That is, in lacking $\hbar$, Eq. (15) can be interpreted classically without having to take the usual classical limit $\hbar \to 0$. However, a lack of the factor $\hbar$ does not necessarily imply a classical result, since the factor is known to cancel from certain other effects that are intrinsically quantum in nature, such as that of coherent light [29].

A. Derivation

Multiplying Eq. (15) by T_k^* gives

$$\frac{1}{2} G_k(t) T_k^* T_k = T_k^* \frac{dT_k}{dt}. \tag{16}$$

Adding this to its complex conjugate gives

$$T_k^* \frac{dT_k}{dt} + T_k \frac{dT_k^*}{dt} \equiv \frac{d}{dt}(T_k^* T_k) = G_k(t) T_k^* T_k. \tag{17}$$

Multiplying by $R_k^*(\mathbf{r}) R_k(\mathbf{r})$ gives

$$\frac{d}{dt}(R_k^* R_k T_k^* T_k) = G_k(t) R_k^* T_k^* R_k T_k.$$

By Eq. (7), this is equivalently

$$\frac{d}{dt}(\psi_k^* \psi_k) = G_k(t) \psi_k^* \psi_k, \quad \text{where } \psi_k \equiv \psi_k(\mathbf{r},t). \tag{18}$$

Equation (18) is equivalently

$$\frac{dp_k}{dt} = G_k(t) p_k, \quad p_k = |\psi_k(\mathbf{r},t)|^2.$$

Next, integrate out over all space $\mathbf{r}$. This gives the marginal probability $p_k(t)$ as obeying

$$\frac{dp_k(t)}{dt} = G_k(t) p_k(t). \tag{19}$$

B. Equation of population growth

Using Eqs. (11) and (12) in Eq. (19) gives

$$\frac{dp_k(t)}{dt} = p_k(t)\left(g_k(t) + \sum_j g_{jk}(t) p_j(t)\right), \quad p_j(t) = \int d\mathbf{r}_j p_j(\mathbf{r},t). \tag{20}$$

The far-right integral represents the probability that the jth particle exists somewhere (anywhere) at an arbitrary time t. This equation is more conveniently expressed as

$$\frac{dp_k(t)}{dt} = p_k(t) G_k(t), \quad k = 1, \ldots, K,$$
$$G_k(t) = g_k(t) + \sum_j g_{jk}(t) p_j(t). \tag{21}$$

This shows that the population $p_k(t)$ of the scattered particles obeys a set of coupled, first-order, generally quadratic, differential equations. It is the main result of the paper.

Equations (21) are classical transport equations. These describe the growth dynamics of a system on the level of its probability density function (PDF) $p_k(t)$, all phase information having been lost in the squaring operation (13). The nature of the transport is defined by the nature of coefficients $g_k(t), g_{jk}(t)$. One example of transport (21) is the physical, quantum-type transport of the $K=2$ relative populations $p_k(t)$ of the energy levels in two-level atoms of the gas of a laser cavity [30]. Here the coefficients $g_k(t), g_{jk}(t)$ are characterized by constant transition rates between the two levels. Another type of transport (21) is biological—the growth and depletion of living populations. An example is ribonucleic acid (RNA) replication [31]. The general biological transport of populations is considered in Sec. VII.

Equation (21) explicitly represents the probabilistic growth history of each single particle named k. It is now shown to represent as well the growth law for classes of particles of types k, now with generally many particles to a class. Basically, this follows because Eq. (21) is linear, as shown next.

C. Transport equations for classes of particles

Since the $p_k(t)$ are probabilities, they must obey the ordinary rules of probability theory [32]. Suppose, now, that particles k_1 and k_1 are of the same type, and consequently are indististinguishable. Call this a particle class k. Then, the PDF for the occurrence of the class k obeys $p_k \equiv p(k_1 \text{ or } k_2) = p_{k1} + p_{k2} = 2p_{k1} = 2p_{k2}$. Also, the single-particle potentials obey $v_{k1} = v_{k2} = v_k$ since each potential is independent of position, and is for the same particle type. Likewise, the two-particle potential interaction v_{jk} of a particle of class k with any particle j in the system now obeys $v_{jk}=v_{jk1}=v_{jk2}$ since both k_1 and k_2 identify the same particle class k independent of position.

The upshot is that if Eq. (21) is expressed for particle k_1, and again for particle k_2, and the two transport equations are added, the result is again Eq. (21), but with a new meaning—the particle k is now any particle of class k. For example $p_k(t)$ represents the PDF for an electron in general. Obviously this approach may be generalized to more than two particles $k_1, k_2, \ldots$. In this way, Eq. (21) now represents transport equations for particle *classes* $k=1,2,\ldots,K$ (renaming K accordingly).

Transport Eq. (21) is reminiscent of the textbook equation of continuity of flow of quantum mechanics [6]. The latter likewise defines the probability current density $dp_k(t)/dt$ in terms of a current density $\mathbf{S}(\mathbf{r},t)$. However, the textbook derivation [6] depends upon the use of a purely real potential function $V(\mathbf{r},t)$, which causes it to drop out of that analysis. By comparison, our potentials (10) are complex, with the result that they stay in, as we see in Eq. (21). Our results are, consequently, quite different, giving transport equations of population change instead of mere continuity of flow.

VII. BIOLOGICAL GROWTH AND DEPLETION

Consider now a specifically biological system (an ecology), consisting of particles that are living creatures. There are K (general) classes of creatures present, and (as in the above) each creature of a class k is indistinguishable from the others of class k. There are generally $n_k(t), k=1,\ldots,K$, creatures of each class present in the system at time t. The relative population at each time is defined as

$$\frac{n_k(t)}{N(t)} \to p_k(t), \quad N(t) \equiv \sum_{k=1}^{K} n_k(t). \tag{22}$$

This $p_k(t)$ is also the relative occurrence of the particle event k at the time t. But this was the significance as well of $p_k(t)$ in Eq. (21). Moreover, Eq. (21) was seen to be a general transport equation, and so should be capable of representing a living, i.e., biological system of population classes $k=1,\ldots,K$ (e.g., $K=2$ in a simple predator-prey system, considered below).

A. Lotka-Volterra equations

In fact it is known that, to a good approximation, biological systems grow according to generalized Lotka-Volterra equations [22,33], and Eqs. (21) have precisely this mathematical form. Thus, if our model assumptions above are to be valid in describing biological systems, what we formerly regarded as *physical* potentials g_k, g_{jk}, G_k must have a corresponding biological interpretation.

In fact, by the standard LV interpretation of Eqs. (21), what we called a “potential” $g_k(t)$ is now the intrinsic (or self-) *fitness* of a species k at the time t, and the interactive potentials $g_{jk}(t)$ become interactive fitness contributions of other species j to that of species k. Analogously for G_k. Note that the fitness is defined as the relative number of offspring per breeding unit (two individuals for sexual breeding), per generation, that later mature to reproduce on their own.

Thus, Eqs. (21), originally developed to define a physical system’s particle growth characteristics via Hartree potentials defines, as well, growth equations governing the populations of a biological system with corresponding fitness coefficients. As we see, this is not mere coincidence, but a direct consequence of the mathematics of the Hartree approach, in the presence of sufficient coherence (as discussed). This Hartree potential-fitness coefficient correspondence is the correspondence principle we sought at the outset of this paper.

At the end of Sec. VII C, the physical realization of one such potential-fitness correspondence is cited.

B. Example 1

As an example, consider a simple system consisting of $K=2$ types of “particle.” There are now two Hartree growth equations (21),

$$\frac{dp_1(t)}{dt} = p_1(t)[g_1(t) + g_{11}(t)p_1(t) + g_{21}(t)p_2(t)],$$

$$\frac{dp_2(t)}{dt} = p_2(t)[g_2(t) + g_{12}(t)p_1(t) + g_{22}(t)p_2(t)]. \tag{23}$$

The coupling coefficients $g_k(t), g_{jk}(t)$ have until now been general. Consider a case where they are constants,

$$g_1(t) = A = \text{const}, \quad g_2(t) = -C = \text{const},$$

$$g_{11}(t) = 0, \quad g_{21}(t) = -B = \text{const}, \quad g_{12}(t) = D = \text{const},$$
$$g_{22}(t) = 0. \tag{24}$$

This represents a scenario of constant rates of particle transport. Using these in Eqs. (23) gives

$$\frac{dp_1(t)}{dt} = Ap_1(t) - Bp_1(t)p_2(t),$$

$$\frac{dp_2(t)}{dt} = -Cp_2(t) + Dp_1(t)p_2(t), \quad A,B,C,D \geqslant 0. \tag{25}$$

In the biological context, these are Lotka-Volterra equations for a simple predator-prey system, where subscript 1 identifies the prey species and 2 the predator. Here A and C represent, respectively, the intrinsic growths of prey and predator (i.e., in absence of predator or prey, respectively); and B, D represent, respectively, the rates at which prey decrease and predators increase due explicitly to the predation (interaction). Hence, the constant coefficients (24) used in the Hartree equations describe both a nonliving particle system of two particle types and a corresponding living system of predator and prey animals.

It may be noticed that in the preceding example the self-interaction terms $g_{11}(t)=g_{22}(t)=0$ are not present. As discussed at the outset, this is a defining property of fermions. Hence, this particular case (24) of coupling constants describes organisms whose population growth statistics are those of fermions. In effect, the LV equations are now of Hartree-Fock form.

C. Example 2

By comparison, a biological case where a self-interaction term $g_{11}(t)$ effectively exists is as follows. With self-interaction terms, the biological particles act effectively like bosons. Consider a predator-prey system that is closed to inputs of population (by migration) but open to energy [34]. Again $p_1(t)$ represents prey population and $p_2(t)$ represents predator. This is described by coupling coefficients

$$g_1(t) = g_1 = \text{const}, \quad g_2(t) = -m_2 = \text{const}, \quad m_2 \geqslant 0,$$

$$g_{11}(t) = -\frac{g_1}{k} = \text{const}, \quad g_{21}(t) = -\frac{l_{12}}{1+\beta p_1(t)},$$

$$g_{12}(t) = \frac{g_{21}}{1+\beta p_1(t)}, \quad g_{22}(t) = 0. \tag{26}$$

Here, g_1 is the intrinsic prey growth rate, m_2 is the intrinsic predator mortality rate, l_{12}=const is the prey loss rate due to predatory feeding, g_{21} is the predator feeding rate, k represents a relative population limit to prey in the absence of a predator and β is a predator satiation term. Using these coefficients in LV equations (23) gives

$$\frac{dp_1(t)}{dt} = p_1(t)\left(g_1 - \frac{g_1}{k}p_1(t) - \frac{l_{12}}{1+\beta p_1(t)}p_2(t)\right),$$

$$\frac{dp_2(t)}{dt} = p_2(t)\left(-m_2 + \frac{g_{21}}{1+\beta p_1(t)}p_1(t)\right). \tag{27}$$

The quadratic term $p_1^2(t)$ in the first growth equation indicates a non-fermion case, i.e., a case of particles analogous to bosons. These are the correct equations for this predator-prey scenario [34].

What physical scattering problem corresponds to the biological case (27)? By the correspondences (10), the constant fitness coefficients g_1, g_2 correspond to constant imaginary potential values v_1, v_2. Also, regarding interactive potentials $g_{jk}(t)$, the constant self-interactive fitness coefficient $g_{11}(t)$ in Eq. (26) corresponds to a constant self-interactive potential v_{11}, while fitness coefficients $g_{21}(t), g_{12}(t)$ in Eq. (26) correspond to interactive potentials $v_{21}(t), v_{12}(t)$ that vary with time as $[1+\beta p_1(t)]^{-1}$. Interestingly, potential functions with such interactive time variations seem physically realizable in the electron impact excitation of neutral and singly ionized nitrogen [35].

D. Applicability

As noted in Sec. II C, the assumption of coherence narrows the applicability of the results to systems whose coherence time is less than the time scale of the scattering process. This point is supplemented by other requirements, discussed next.

For these LV growth effects to have real biological consequence, as with any living system the scatter must be open to the outside environment. Further, since it is difficult to envision a *living* atom, the system probably would be mesoscopic, i.e., have dimensions somewhere between the atomic and macroscopic domains.

In fact, the subject of coherence for mesoscopic systems is currently the subject of much study. Generally, the coherence time depends upon the parameters of the system and the reservoir to which it is coupled. Its values can range over many orders from very small (currently nonmeasureable) for macrosystems, to very large values for almost isolated elementary particles [36]. For Josephson flux qubits, coherence times of up to several microseconds have been found [37]. Coherence times going inversely with the temperature T, as $T^{-\gamma}$, $1 \lesssim \gamma \lesssim 2$, have been found [38] to characterize quantum dots. Thus, for small enough temperature the coherence time of this system could be considerable, and our approach realized. Other investigators [39] computer simulate AlGaAs/GaAs resonant transistor diodes via the Hartree approach, where the Hartree potential that is used is determined self-consistently with the SWE through solution of a Poisson equation. The results qualitatively agree with experiment, potentially validating this application of the Hartree approach.

Of course a physically deeper question, not addressed in this paper, is how generally a classical world emerges from quantum theory. See [40,41].

VIII. CONCLUSIONS

We asked whether population transport properties can be shown to derive theoretically from quantum dynamics. It was shown that the answer is yes.

The same Hartree approach that models a many-body problem in the Schrödinger picture can, in the absence of decoherence, also describe the growth of a corresponding "many-populations" system of biology. In fact, every interactive potential of the Hartree many-body system has a counterpart particle-interactive fitness component in the corresponding LV biological system.

The correspondence, an outgrowth of corresponding terms in the large parentheses of Eqs. (4) and (20), is that interactive potentials in the latter act theoretically like interactive fitnesses in the former. This is the correspondence principle we sought at the outset. Two specific predator-prey ecologies were emulated in this way by corresponding particle scattering cases.

A more accurate description of biological evolution incorporates evolutionary tactics c_k, $k=1,\ldots$ in its makeup. In fact, it has been shown [42] that tactics can be so incorporated, directly into the fitness values $g_k(c_k,t), g_{jk}(c_k,t)$ of the populations. Hence, the correspondences between Eqs. (4) and (20) imply, further, that even this refinement on evolutionary theory can be accommodated by potentials. These are now of the same functional forms $v_k(c_k,t), v_{jk}(c_k,t)$ as the potentials.

Thus, the experimenter with (i) control over the potentials in a physical scattering system and (ii) able to operate within the system coherence time (as discussed in Sec. VII D) can, by the Hartree approximation, (iii) force its particles to obey the same growth equations as the population components of a corresponding biological system. In addition, the biological system fitness values can be realistically affected by both reproductive choice and evolutionary tactics c_k.

Does this mean that macroscopic biological growth is basically quantum mechanical in origin? Certainly its dynamics follow a wave equation, but it remains to be shown that the other quantum properties mentioned at the outset also hold. This question also begs the following questions.

IX. SPECULATIONS

We showed that biological growth can, in principle, result from physical Hartree scattering with appropriate potentials and sufficiently large coherence time. On this basis, life could conceivably have originated from such a scattering process. However, although a living system must necessarily obey general Lotka-Volterra growth equations [17,22,33,36] the condition is not sufficient. Certain nonliving systems also obey Lotka-Volterra-type equations, albeit with a very limited number of interaction fitnesses. An example is the two populations in a lasing medium [30].

What would uniquely imply that a system is living, aside from obeying LV growth, is that it also obeys other key biological effects such as pursuing and absorbing nutrients, and passing on genes to the next generation. Now, as we discussed, at least some of these are fixed by factors c_k that influence fitnesses $g_k(c_k,t), g_{jk}(c_k,t)$. And, as we see in the preceding section, such fitnesses can be emulated one-to-one by corresponding Hartree scattering potentials. Thus, other properties of life can potentially be emulated, as well, by Hartree scatter. Furthermore, it seems reasonable that the more such properties can be emulated the more probable it is that the emulation becomes reality, i.e., the system is actually alive. However, this assumes that the functional forms $g_k(c_k,t), g_{jk}(c_k,t)$ of the time-dependent population fitnesses are known in the first place. At present such detailed knowledge does not appear to be available, although they might conceivably have occurred naturally some three billion years ago.

A preliminary version of this quantum approach to biological equations of growth may also be found in [17].

ACKNOWLEDGMENTS

We thank Professor T. L. Vincent for his ever-perceptive and creative remarks on questions of biological growth. This research was supported by the Technology Research Initiative Fund (2006–2008) of the Arizona Board of Regents, Phoenix, Arizona.

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